CONCERNING THE CUT POINTS OF CONTINUA*

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In this paper propositions will be established concerning the collection of all the cut points of a given plane† continuum. It will be shown that there does not exist an uncountable collection of mutually exclusive subcontinua of a given continuum M each of which contains at least one cut point of M. With the aid of this fundamental theorem it is shown, among other things, that the set of all the cut points of a continuum M which are irregular points of M is necessarily countable and, indeed, that all save a countable number of the cut points of any continuum M are points of Menger order two of M.

1. Preliminary notions. Intervals and segments of a continuum

Definitions. The points, or point sets, A and B of a continuum M are said to be separated in M by the point X of M provided that $M-X$ is the sum of two mutually separated point sets $M_a(X)$ and $M_b(X)$ containing A and B respectively. The notation $K(A, B)$ will be used to denote the set of all points of M which separate A and B in M.

Let A and B be any two points of a continuum M and X and Y any two distinct points of $K(A, B)$. Then X will be said to *precede* Y on M in the order from A to B provided that X belongs to the set $M_a(Y)$; X will be said to follow Y provided that X belongs to the set $M_b(Y)$. It is easily seen that if X precedes Y, then Y follows X. If X precedes Y, then by the *interval of M from X to Y* (or from Y to X), or the *interval XY* (or YX) *of M*, will be meant the set of points $[M_b(X)+X]\cdot[M_a(Y)+Y]$, and by the *segment* of M from X to Y will be meant the set of points $M_b(X)\cdot M_a(Y)$. If X follows Y, then since Y precedes X, by the interval XY of M is meant the set of points $[M_b(Y)+Y]\cdot[M_a(X)+X]$, and the segment XY (or YX) of M means the set of points $M_a(X)\cdot M_b(Y)$. The notation $I(X, Y)$ for the interval of M from X to Y and $S(X, Y)$ for the segment of M from X to Y will be used in this paper. Clearly $S(X, Y)=I(X, Y)-(X+Y)$.

The following two propositions can be proved with little difficulty from the above definitions.

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† Quite recently I have established some very general results on the cuttings of continua, as a consequence of which it follows that all the theorems in this paper hold true in a euclidean space of n dimensions, and nearly all of them hold in any separable metric space which is locally compact. See my forthcoming paper *Concerning collections of cuttings of connected point sets.*

THEOREM 1. *Let A and B be any two points of a continuum M, and let X and Y be any two distinct points of $K(A, B)$, where X precedes Y. Then M is the sum of three continua $M_a(X)+X$, $I(X, Y)$, and $M_b(Y)+Y$, where $[M_a(X)+X]\cdot I(X, Y)=X$, $[M_b(Y)+Y]\cdot I(X, Y)=Y$, and $[M_a(X)+X]\cdot[M_b(Y)+Y]=0$.*

THEOREM 2. *Using the same notation as in Theorem 1, suppose that each of the sets $M_a(X)$, $M_b(X)$, $M_a(Y)$, and $M_b(Y)$ is connected. Then if $S(X, Y)$ contains a point of $K(A, B)$, $S(X, Y)$ is connected.*

2. THE PRINCIPAL RESULTS

THEOREM 3. *If H is any uncountable subset of the set G of all the cut points of a continuum M, then there exist two points A and B of H which are separated in M by uncountably many of the points of H.**

There exists a countable subset D of M such that every point of M is a limit point of D. Let F be the collection of all possible pairs of points of D. Then F is countable. For each pair of points u, v of F, let $K_h(u, v)$ denote the set of all points of H that separate u and v in M. Then $H=\sum_F K_h(u, v)$. For let X be a point of H. Then since X is a cut point of M, $M-X=M_1+M_2$, where M_1 and M_2 are mutually separated point sets. Since every point of M is a limit point of D, it follows that M_1 contains a point U of D and M_2 contains a point V of D. Then X separates U and V in M, and hence it belongs to $K_h(U, V)$. Therefore $H=\sum_F K_h(u, v)$; and since F is countable and H is uncountable, it follows that there exists at least one pair P, Q of F such that $K_h(P, Q)$ is uncountable.

Let E denote the set of points $K_h(P, Q)$. Then since E is uncountable, it contains a point A which is a point of condensation of E, i.e., every vicinity of A contains uncountably many points of E. Now for at least one of the sets $M_p(A)$ and $M_q(A)$, say for $M_q(A)$, it is true that A is a point of condensation of the subset of E which belongs to $M_q(A)$. Let B be a point of E belonging to $M_q(A)$. Then by Theorem 1 it readily follows that the segment AB of M must contain an uncountable subset E_0 of E. Let X be any point of E_0. Then since X belongs to $S(A, B)$, it follows by Theorem 1 that neither of the continua $M_p(A)+A$ and $M_q(B)+B$ contains the point X. And as these continua contain the points P and Q respectively, therefore they are subsets of $M_p(X)$ and $M_q(X)$ respectively. Hence A and B are separated in M by each point X of E_0, and as E_0 is an uncountable subset of H, the truth of Theorem 3 is therefore established.

* The referee has kindly called my attention to the fact that Theorems 3 and 4 of this paper are valid in any number of dimensions, since my proofs of these theorems make no use of the dimensionality of the space in which the point sets lie.

THEOREM 4. *Let G denote the set of all the cut points of a continuum M. Then for all, save possibly a countable number, of points X of G it is true that $M-X$ is the sum of two mutually separated and connected point sets.**

Suppose, on the contrary, that there exists an uncountable subset D of G such that if X is any point of D, then $M-X=M_1(X)+M_2(X)$, where $M_1(X)$ and $M_2(X)$ are mutually separated point sets and at least one of them is not connected. Then by Theorem 3 there exist two points A and B of D which are separated in M by each point of an uncountable subset H of the set D. For each point X of H, $M-X=H_a(X)+H_b(X)+H_0(X)$, where $H_a(X)$, $H_b(X)$, and $H_0(X)$ are mutually separated point sets, and $H_a(X)$ and $H_b(X)$ contain A and B respectively.

Now for each point X of H, the set H, and indeed the set $K(A, B)$, is a subset of the set of points $H_a(X)+H_b(X)+X$, because this set of points is connected and contains both A and B and therefore must contain every point of M which separates A and B in M. It follows that if X and Y are distinct points of H, the set of points $H_0(Y)$ belongs to the set $H_a(X)+H_b(X)$, and the set $H_0(X)$ belongs to the set $H_a(Y)+H_b(Y)$; and therefore the sets $H_0(X)$ and $H_0(Y)$ are mutually exclusive. By the Zermelo postulate there exists a set of points E such that for each point X of H there is exactly one point P_x of E which belongs to the set of points $H_0(X)$, and for each point P_x of E there is exactly one point X of H such that the set $H_0(X)$ contains the point P_x. Since H is uncountable, E must be uncountable. Hence there exists a point P_z of E which is a limit point of E. But there exists a point Z in H such that P_z belongs to the set $H_0(Z)$, and $E-P_z$ belongs to the set of points $H_a(Z)+H_b(Z)$. Clearly this is impossible, because $H_a(Z)$, $H_b(Z)$, and $H_0(Z)$ are mutually separated point sets. Thus the supposition that Theorem 4 is not true leads to a contradiction.

Theorem 4 can also be proved very readily with the aid of a statement given, without proof, by C. Zarankiewicz in his paper *Sur les points de division dans les ensembles connexes.*†

THEOREM 5. *Let M be any continuum whatever. Then there does not exist an uncountable collection of mutually exclusive subcontinua‡ of M each of which contains at least one cut point of M.*

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* Since this paper was submitted to the editors, a result somewhat more general than Theorem 4 has been published by Kuratowski and Zarankiewicz; cf. *A theorem on connected point sets,* Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 571–575.

† Cf. Fundamenta Mathematicae, vol. 9, pp. 124–171, footnote to p. 137.

‡ In this paper the hypothesis that a set is a continuum is understood to imply that the set contains more than one point.

Before proceeding with the proof of Theorem 5, I shall establish two lemmas which will be needed in the proof.

LEMMA 1. *If C is any subcontinuum of a continuum M, then not more than a countable number of the bounded maximal connected subsets of $M-C$ can contain cut points of M.*

Suppose, on the contrary, that there exists an uncountable collection G of bounded maximal connected subsets of $M-C$ each of which contains at least one cut point of M. From each element g of G select exactly one cut point P_g of M, and let K denote the set of all points $[P_g]$ thus selected. For each point P_g of K, $M-P_g=M_c(P_g)+M_0(P_g)$, where $M_c(P_g)$ and $M_0(P_g)$ are mutually separated sets and $M_c(P_g)$ contains the continuum C. Let H denote the sum of all the point sets of the collection G. Then since each element of G has* at least one limit point in C, it follows that for each point P_g of K, the set of points $H-g$ belongs to the set $M_c(P_g)$. By the Zermelo postulate, there exists a set of points E such that for each point P_g of K there is exactly one point Q_g of E which belongs to the set of points $M_0(P_g)$, and for each point Q_g of E there is exactly one point P_g of K such that the set of points $M_0(P_g)$ contains the point Q_g. It is readily seen that no two of the sets $[M_0(P_g)]$ can have a common point. Therefore, as H is uncountable, both K and E must be uncountable. Hence E contains a point Q_i which is a limit point of E. But clearly this is impossible, because there exists a point P_i in K such that Q_i belongs to the set $M_0(P_i)$ and $E-Q_i$ belongs to the set $M_c(P_i)$, and $M_0(P_i)$ and $M_c(P_i)$ are mutually separated sets by definition. This contradiction proves Lemma 1.

LEMMA 2. *If M is any continuum, C is any simple closed curve, and G_0 denotes the collection of all the maximal connected subsets of M contained in C plus its interior, then not more than a countable number of elements of G_0 can contain cut points of M which lie within C.*

Suppose, on the contrary, that G_0 contains an uncountable subcollection G each element g of which contains at least one cut point P_g of M lying within C.

(1) Not more than a countable number of elements g of G can have the property that $M-P_g$ can be expressed as the sum of two mutually separated sets one of which lies wholly within C. For suppose the contrary is true. Then, clearly, for each such element g of G, the point P_g is a cut point of the continuum $M+C$. But since each such point P_g lies on a bounded maximal

* See my paper *Concerning point sets which can be made connected by the addition of a simple continuous arc*, these Transactions, vol. 29 (1927), footnote to p. 754.

connected subset of $(M+C)-C$, and no two different points $[P_g]$ can lie on the same maximal connected subset of $(M+C)-C$, this is contradictory to Lemma 1.

(2) Let G_1 denote the collection of all those elements g of G such that however $M-P_g$ be expressed as the sum of two mutually separated point sets $M_1(P_g)$ and $M_2(P_g)$, each of these sets must contain at least one point of C. It follows by (1) that G_1 must be uncountable. It is clear that if g is any element of G_1 and we let $g_1(P)=g\cdot M_1(P_g)$ and $g_2(P)=g\cdot M_2(P_g)$, then $g-P_g=g_1(P)+g_2(P)$, and $g_1(P)$ and $g_2(P)$ are mutually separated point sets each of which contains at least one point on C. Now C is not a subset of M, for if it were, then G_0 would have only one element. Let the maximal segments of $C-M\cdot C$ be denoted by $S_1, S_2, S_3, \cdots$. Let g be any element of G_1. Denote the end points of S_1 by A and B, and let AOB be the arc of C from A to B which does not contain S_1. On AOB, in the order from A to B, let X_g be the first point belonging to g. Then X_g belongs either to $g_1(P)$ or to $g_2(P)$, say to $g_1(P)$. Then on the arc X_gB of AOB, in the order from X_g to B, let Y_g be the first point belonging to $g_2(P)$. It is easy to see that the segment X_gY_g of the arc AOB contains at least one segment S_i of the sequence $S_1, S_2, S_3, \cdots$. Let A_i and B_i denote the end points of S_i, where A_i precedes B_i on AOB in the order from A to B. Then if T_1 and T_2 denote the arcs AA_i and B_iB respectively of AOB, $g_1(P)$ has at least one point on one of these arcs and $g_2(P)$ has at least one point on the other. Thus it is seen that for each element g of G_1 there exist two segments of the sequence $S_1, S_2, S_3, \cdots$ such that $g_1(P)$ has a point on one of the complementary arcs of C of these two segments and $g_2(P)$ has a point on the other. Then since the number of segments $S_1, S_2, S_3, \cdots$ is countable, and G_1 is uncountable, there exist two segments S_k and S_j of the sequence $S_1, S_2, S_3, \cdots$ such that if T and L denote the complementary arcs of C, then for uncountably many elements g of G_1 it is true that $g_1(P)$ has at least one point on one of the arcs T and L, and $g_2(P)$ has at least one point on the other. Let G_2 denote the collection of all those elements of G_1 which have this property.

(3) There are not more than a countable number of elements g of G_2 such that $T+L$ contains as many as three points of g. For suppose the contrary is true. Then there exists an uncountable subcollection E of G_2 such that each element e of E has at least two points on one of the arcs T and L, say on T, and at least one point on L. Let A and B denote the end points of T, where A is an end point of S_k, and for each element e of E, let A_e and B_e respectively denote the first and last points on T in the order from A to B which belong to e. Then for no two elements e and f of E can the segments

A_eB_e and A_fB_f of T have a common point. For suppose they do. Then either the segment A_eB_e contains a point of f or the segment A_fB_f contains a point of e. The two cases are alike, so let us suppose that the segment A_eB_e contains a point X of f. Now L contains at least one point Y of f. There exists an arc XZY from X to Y which lies, except for the points X and Y, wholly without C. It is readily seen that the segments S_k and S_j belong to different complementary domains R_k and R_j respectively of the continuum $f+XZY$. Since A_e can be joined to A and B_e to B by arcs lying, except for their end points, in the exterior of C and containing no point whatever of XZY, it follows that A_e must belong to R_k and B_e to R_j. Then since e is connected, it must have at least one point in common with $f+XZY$. But as the segment XZY lies without C, e can have no point in common with it. Hence e must have a point in common with f. This is contradictory to the fact that e and f are different maximal connected subsets of M lying in C plus its interior. Therefore no two segments A_eB_e and A_fB_f can have a common point. But since E is uncountable, then the number of these segments must be uncountable. Clearly this is impossible. This contradiction proves the statement made in the first sentence of this paragraph.

(4) It follows from (3) that there exists an uncountable subcollection G_3 of elements of G_2 such that if e is any element of G_3, then $e_1(P)$ has exactly one point X_e on T and $e_2(P)$ has exactly one point Y_e on L. But since G_3 is uncountable, it contains an element g which contains the sequential limiting set H of some infinite sequence V of elements of G_3. Then H is connected and contains at least one point on each of the arcs T and L. But since g has in common with $T+L$ only the points X_g and Y_g, H must contain both X_g and Y_g. Now some infinite subsequence V_1 of V has the property that all of its elements belong to one of the sets $M_1(P_g)$ and $M_2(P_g)$, say to $M_1(P_g)$. But H is also the limiting set of the sequence V_1, and the point Y_g belongs to the set $M_2(P_g)$. Hence $M_2(P_g)$ contains a limit point of $M_1(P_g)$, contrary to the fact that $M_1(P_g)$ and $M_2(P_g)$ are mutually separated point sets. Thus the supposition that Lemma 2 is not true leads to a contradiction.

Proof of Theorem 5. Suppose, contrary to Theorem 5, that there exists an uncountable collection G of mutually exclusive subcontinua of M each of which contains at least one cut point of M. From each element g of G select exactly one cut point P_g of M, and let N denote the set of all points $[P_g]$ thus selected. Since N is uncountable, there exists a square S which encloses uncountably many points of N. Let N_1 be the set of all those points of N which are within S. There exists a positive number d and an uncountable subset N_2 of N_1 such that if G_2 denotes the collection of all those elements of G which contain a point of N_2, then each element of G_2 is of diameter greater than $4d$. Let us divide S plus its interior into a finite collection E of squares plus their interiors by lines parallel and perpendicular to the bases of S in such a way that the diameter of each square of E is less than d. Then either some side T of one of the squares of E contains uncountably many points of N_2, or else some square of E encloses uncountably many of these points. In either case it is readily seen that there exists a rectangle R_1 of diameter less than $2d$ which encloses uncountably many of the points of N_2. There exists a rectangle R_2 concentric with and within R_1 which also encloses an uncountable subset N_3 of N_2.

Let F denote the collection of all the maximal connected subsets of M contained in R_2 plus its interior. Since N_3 is uncountable and each point of N_3 belongs to some element of F, and since, by Lemma 2, not more than a countable number of elements of F can contain points of N_3, it follows that some element K of F must contain an uncountable subset N_4 of N_3. It follows by a theorem of R. L. Moore's* that uncountably many points of N_4 must be cut points also of the continuum K. Let N_5 be the set of all those points of N_4 which are cut points of K. By Theorem 3, there exist two points A and B of N_5 which are separated in K and also in M by uncountably many points of N_5. Let N_6 be the set of all those points of N_5 which separate A and B in K and also in M, and let G_6 be the collection of all those elements of G_2 which contain a point of N_6. Since each element of G_6 is of diameter greater than $4d$, and R_1 is of diameter less than $2d$, each element of G_6 must contain a point of R_1 and also a point of R_2. For each element g of G_6, the maximal connected subset H_{0g} of g which contains the point P_g and lies within R_1 has at least one limit point A_{P_g} which belongs to R_1. Let H_{P_g} denote the continuum obtained by adding to H_{0g} all of its limit points. Now since A and B belong to N_5, the elements g_a and g_b of G_2 which contain A and B respectively contain subcontinua H_a and H_b respectively which contain A and B respectively and at least one point U and V respectively on R_1 but which lie wholly in R_1 plus its interior. It follows by Theorem 4 that N_6 contains an uncountable subset N_7 such that if X is any point of N_7, then $M-X=M_a(X)+M_b(X)$, where $M_a(X)$ and $M_b(X)$ are mutually separated and connected point sets containing A and B respectively, and also $K-X=K_a(X)+K_b(X)$, where $K_a(X)$ and $K_b(X)$ are connected sets belonging to $M_a(X)$ and $M_b(X)$ respectively. Let E_0 denote the collection of all the continua $[H_x]$ selected above which correspond to points X of N_7. At least one of the arcs of R_1 from U to V must contain uncountably many points

* R. L. Moore, *Concerning the cut points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

A_x which correspond to points X of N_7 and belong to continua H_x of the collection E_0; let UOV denote one which does, and let N_8 be the set of all those points of N_7 such that the corresponding point A_x belongs to the arc UOV.

Now if X and Y are two points of N_8 such that X precedes Y on K in the order from A to B (see §1), I shall show that the point A_x precedes the point A_y on the arc UOV in the order from U to V. Suppose, on the contrary, that A_y precedes A_x on UOV in the order from U to V. Then U and A_x separate the points V and A_y on R_1, and if UWA_x is an arc from U to A_x which lies, except for its end points, in the exterior of R_1, then it is easy to see that the points A_y and V lie in different complementary domains of the continuum $UWA_x+H_a+K_a(X)+H_x$. Then since the continuum $H_b+K_b(Y)+H_y$ contains both A_y and V, therefore it must contain at least one point in common with $UWA_x+H_a+K_a(X)+H_x$. This is impossible, for no one of the sets H_b, $K_b(Y)$, and H_y can have a point in common with any one of the sets UWA_x, H_a, $K_a(X)$, and H_x. Thus the supposition that A_x does not precede A_y on the arc UOV in the order from U to V leads to a contradiction.

Now since N_8 is uncountable, the set of corresponding points $[A_x]$ is uncountable. Accordingly there exists one of these points A_z which is a limit point of a subset S_1 of the remaining ones belonging to the arc UA_z of UOV and also of a subset S_2 of the remaining ones belonging to the arc A_zV of UOV. Now the point A_z belongs either to $M_a(Z)$ or to $M_b(Z)$, say to $M_a(Z)$. But this is impossible, since A_z is a limit point of S_2, and if A_x is any point of S_2, it was shown above that the point X follows the point Z on M in the order from A to B (for the order on M and on K is the same), i.e., X belongs to $M_b(Z)$; and thus A_z is a limit point of $M_b(Z)$, which is absurd. Likewise we arrive at a contradiction if we suppose that A_z belongs to $M_b(Z)$. Thus the supposition that Theorem 5 is not true leads to a contradiction and the theorem is therefore established.

Theorem 6. *If E is any subset of the set of all the cut points of a continuum M, then there are not more than a countable number of points X of E such that X belongs to some subcontinuum of M which contains no other point of E.*

Suppose, on the contrary, that there exists an uncountable subset D of E such that each point X of D belongs to some subcontinuum N_x of M containing no other point of E. Then by Theorems 3 and 4 it follows that there exist two points A and B of M and an uncountable subset F of D such that if X is any point of F, X separates A and B in M and furthermore each of the sets $M_a(X)$ and $M_b(X)$ is connected.

Now for not more than a countable number of points X of F is it true that there exists some other point Y of F such that the segment XY of M contains no point whatever of F. For let Q be the set of all points X of F such that there exists a point Y of F such that $S(X, Y)$ contains no point of F, and suppose, contrary to the statement just made, that Q is uncountable. It follows that there exists an uncountable collection of the segments $[S(X, Y)]$ no two of which are identical and no one of which contains a point of F. Let $S(X_1, Y_1)$ and $S(X_2, Y_2)$ be any two of these segments, where X_1 precedes Y_1 and X_2 precedes Y_2 on M in the order from A to B. Then $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are mutually exclusive. For suppose they have a point P in common. One of these segments is not a subset of the other one, for they are not identical. Suppose $S(X_2, Y_2)$ is not a subset of $S(X_1, Y_1)$. Let K denote the set of points common to $S(X_1, Y_1)$ and $S(X_2, Y_2)$. Now not both of the points X_1 and Y_1 can be limit points of K; for if they were, they would both belong to $I(X_2, Y_2)$ and at least one of them would have to belong to $S(X_2, Y_2)$, contrary to the fact that $S(X_2, Y_2)$ contains no point of F. The two cases are alike, so let us suppose X_1 is not a limit point of K. Then it is easy to see that K is closed except possibly for the point Y_1. Since K belongs to $S(X_1, Y_1)$, it must belong to $M_a(Y_1)$. But $M_a(Y_1)$ is connected; and since $M_a(Y_1)-K$ contains no limit point of K, therefore K must contain at least one limit point of $M_a(Y_1)-K$. But clearly this is impossible, since by Theorem 1 it follows that $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are open subsets of M. Thus the supposition that $S(X_1, Y_1)$ and $S(X_2, Y_2)$ are not mutually exclusive leads to a contradiction. Hence no two of the segments of the collection $[S(X, Y)]$ can have a common point. But since, by Theorem 1, each of these segments is an open subset of M and since $[S(X, Y)]$ is an uncountable collection, this clearly is impossible. Thus the supposition that Q is uncountable leads to a contradiction.

Now let H denote the set of points $F-Q$. Then since F is uncountable and Q is countable, H must be uncountable. By hypothesis, for each point X of H there exists a subcontinuum N_x of M which contains X but which contains no other point whatever of E. Now if X and Y are any two distinct points of H, the continua N_x and N_y can have no point in common. For suppose they do have a point P in common. The point Y belongs either to $M_a(X)$ or to $M_b(X)$, say to $M_b(X)$. Since X and Y belong to H, the segment XY of M must contain at least one point O of F, and O separates A and B in M. But the set of points $M_a(X)+M_b(Y)+N_x+N_y$ is connected and contains both A and B and does not contain the point O. Clearly this is impossible. Thus the supposition that N_x and N_y have a point in common leads to a contradiction. Now since H is uncountable, the collection of continua $[N_x]$ must be uncountable. But no two of these continua have a common point, and each of them contains at least one cut point of M. This is contradictory to Theorem 5. Thus the supposition that Theorem 6 is not true leads to a contradiction, and the theorem is proved.

3. The order and regularity of the cut points of a continuum

Definitions. The point P of a continuum M is said to be a point of *Menger order n of M** provided that for each positive number ϵ there exists a domain U containing P and of diameter less than ϵ and whose boundary has not more than n points in common with M, and furthermore n is the smallest positive integer such that this property is preserved. A point P of a continuum M will be called a regular or an irregular point of M according as M is or is not connected im kleinen at P.

Theorem 7. *Let G denote the set of all the cut points of any continuum M. Then all save possibly a countable number of the points of G are points of Menger order two of M.*

Suppose, on the contrary, that there exists an uncountable subset H of G no point of which is a point of order two of M. Then by Theorem 3, there exist two points A and B of H which are separated in M by each point of an uncountable subset D of H. It follows by Theorem 6 that there exists at least one point P of D such that every subcontinuum of M which contains P must contain at least one point of D distinct from P. It is easy to see that each of the sets $M_a(P)$ and $M_b(P)$ must be connected and that P is a limit point of a subset D_1 of D belonging to $M_a(P)$ and also of a subset D_2 of D belonging to $M_b(P)$.

Let K_1 denote the continuum $M_a(P)+P$ and K_2 the continuum $M_b(P)+P$. There exists a sequence of points $X_1, X_2, X_3, \cdots$, belonging to D_1 and having P as its sequential limit point and such that for each positive integer n, X_n precedes X_{n+1} on M in the order from A to B. Now let us consider the sequence of intervals $I(X_1, X_2), I(X_2, X_3), I(X_3, X_4), \cdots$, of M. R. G. Lubben has shown† that if N denotes the sequential limiting set of this sequence, then N exists and is closed and connected. Clearly N contains the point P. It must consist of P alone. For suppose it does not. Then N is a subcontinuum of K_1 which contains P. It is readily seen that N cannot contain any point of the set of points

* Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272–306.

† See an abstract of his paper *Concerning limiting sets* in the Bulletin of the American Mathematical Society, vol. 32 (1926), p. 14. See also Zarankiewicz, loc. cit.

$$L = M_a(X_1) + \sum_{i=1}^{\infty} I(X_i, X_{i+1}).$$

But L is connected and P is a limit point of L. Hence K_2+L is connected and contains both A and B but contains in common with N only the point P. Hence N contains P but contains no other point of D. This contradicts our definition of the point P. Thus it follows that N is identical with P.

Now let ϵ be any positive number. Then since the sequential limiting set of the sequence of intervals $I(X_1, X_2)$, $I(X_2, X_3)$, $\cdots$, consists of just the point P, and since P is the sequential limit point of the sequence of points $X_1, X_2, X_3, \cdots$, it is readily seen that there exists a positive integer k such that if I denotes the set of points $\sum_{i=k-1}^{\infty} I(X_i, X_{i+1})$, every point of I is at a distance less than $\epsilon/4$ from P. Now by Theorem 1, $I(X_k, P)$ is connected, and it contains X_k. It does not contain X_{k-1}, because X_{k-1} precedes X_k on M in the order from A to B. Hence $I(X_k, P)$ must be a subset of $M_b(X_{k-1})$, and since it is a subset also of K_1, it must be a subset of $I+P$. But $I+P$ is of diameter less than $\epsilon/2$. Hence $I(X_k, P)$ is of diameter less than $\epsilon/2$. In an entirely similar manner it is shown that D_2 contains a point Y_k such that $I(P, Y_k)$ is of diameter less than $\epsilon/2$. Then $I(X_k, Y_k)$ contains P and is of diameter less than ϵ. And since the maximal connected subset of $S(X_k, Y_k)$ which contains P must have both X_k and Y_k as limit points, it is easily seen that there exists a domain of diameter less than ϵ which contains P and whose boundary has in common with M just the points X_k and Y_k. Hence P is a point of Menger order two of M. But by supposition P belongs to H, and no point of H is a point of Menger order two of M. Thus the supposition that Theorem 7 is not true leads to a contradiction.

THEOREM 8. *If A and B are any two points of a continuum M, and P is any point belonging to $K(A, B)$ and having the property that every subcontinuum of M which contains P contains at least one point of $K(A, B)$ distinct from P, then P is a point of Menger order two of M.*

The argument given to prove Theorem 7 also proves Theorem 8.

THEOREM 9. *If P is any cut point of a continuum M which is a point of Menger order two of M, then $M-P=M_1(P)+M_2(P)$, where $M_1(P)$ and $M_2(P)$ are connected point sets and P is a point of order one (i.e., an end point in the Menger sense) of each of the continua $M_1(P)+P$ and $M_2(P)+P$.*

THEOREM 10. *If P is any cut point of a bounded continuum M which is a point of Menger order two of M, then P belongs to the boundary of just one complementary domain of M.*

By a theorem of R. L. Moore's*, P belongs to the boundary of at least one complementary domain of M. Suppose, contrary to this theorem, that P belongs to the boundary of two complementary domains R_1 and R_2 of M. Then there exists† a simple closed curve J which encloses P and is of diameter less than half the diameter of R_1 and also less than half the diameter of R_2, and which has in common with M just two points A and B. Then of the two arcs of J from A to B, one of them, say AXB, must belong to R_1, and the other, AYB, must belong to R_2. But from Theorem 9 and a theorem of R. L. Moore's‡ it follows that there exists a simple closed curve C enclosing one of the sets $M_1(P)$ and $M_2(P)$ and not the other (where $M-P=M_1(P)+M_2(P)$, as in Theorem 9), and containing in common with M only the point P. It is easy to see that C must contain at least one point U of AXB and at least one point V on AYB. Then if t denotes the arc of M from U to V which does not contain P, then t contains no point of M. But this is impossible, because U and V belong to different complementary domains of M. Thus the supposition that Theorem 10 is not true leads to a contradiction.

THEOREM 11. *If G denotes the set of all the cut points of a continuum M, then all save possibly a countable number of the points of G are regular points of M.*

Theorem 11 is a direct consequence of Theorem 7 and of Menger's§ theorem that a continuum M is connected im kleinen at each of its points which is a point of finite order of M.

THEOREM 12. *If D denotes the collection of all the cut points $[P]$ of a continuum M such that P is an irregular point of some subcontinuum of M, then D is countable.*

Theorem 12 follows at once from Theorem 7 and the fact that any point P of Menger order two of a continuum M is a regular point of every subcontinuum of M which contains P.

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* R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213.

† This follows readily from a theorem of R. G. Lubben's. See the abstract of his paper in the Bulletin of the American Mathematical Society, vol. 32 (1926), p. 114.

‡ R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476, Theorem 2. See also R. G. Lubben, loc. cit.

§ Cf. K. Menger, loc. cit.

4. Miscellaneous results

In this section some theorems will be stated with little or no proof given. These theorems, in general, either follow readily from the above propositions already proved or else they can be proved readily using methods similar to those employed in the above proofs.

THEOREM 13. *Let G denote the set of all the cut points of any continuum M. Then G is a subset of the sum of a countable number of bounded subcontinua of M each of which is irreducible between some pair of points.*

Theorem 13 can be proved with the aid of Lemma 2 to Theorem 5 and methods similar to those used in the proof of Theorem 4, together with the theorem of Janiszewski's* that every two points of a bounded continuum M lie together in a subcontinuum of M which is irreducible between those two points.

THEOREM 14. *The set G of all the cut points of a continuous curve M (bounded or not) is a subset of the sum of a countable number of simple continuous arcs which belong to M.*

THEOREM 15. *If N is any subcontinuum of a continuum M which is irreducible between some pair of points A and B of M, then all save possibly a countable number of the cut points of M which belong to N must separate A and B in M.*

COROLLARY. *If t is any simple continuous arc belonging to a continuous curve M, and A and B denote the end points of t, then all save possibly a countable number of the cut points of M which lie on t separate A and B in M.*

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* S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912).

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